Solutions: Core Mathematics 1
January 2008

1

$$\frac{4}{3-\sqrt{7}} = \frac{4}{3-\sqrt{7}} \times \frac{3+\sqrt{7}}{3+\sqrt{7}}$$

$$= \frac{12+4\sqrt{7}}{9-3\sqrt{7}+3\sqrt{7}-7}$$

$$= \frac{12+4\sqrt{7}}{2}$$

$$= 6+2\sqrt{7}$$

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2 (i) The equation of a circle centre (0, 0) radius r is $x^2 + y^2 = r^2$ (Learn this!)
So if the radius is 7, the equation is $x^2 + y^2 = 49$

(ii) There are various ways to tackle this question.
The equation of the circle is $x^2 + y^2 - 6x - 10y - 30 = 0$

First we could try to write this in an alternative form:

$$x^2 - 6x = (x-3)^2 - 9$$
$$y^2 - 10y = (y-5)^2 - 25$$

So the equation of the circle can be expressed as

$$(x-3)^2 - 9 + (y-5)^2 - 25 - 30 = 0$$

or $(x-3)^2 + (y-5)^2 = 64$

Therefore the radius is 8 (the square root of 64).

Note: Here you need to know that the equation of a circle centre (a, b) and with radius r is:
$(x-a)^2 + (y-b)^2 = r^2$

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3 The right hand side can be expanded out:

$$a(x+3)^2 + c = a(x+3)(x+3) + c = a(x^2 + 6x + 9) + c$$

(Remember that a bracket is squared by multiplying it by itself.)
So

$$a(x+3)^2 + c = ax^2 + 6ax + 9a + c\,.$$

Compare this with the left hand side:

$$3x^2 + bx + 10 = ax^2 + 6ax + 9a + c$$

Comparing coefficients of x^2: 3 = a

Comparing coefficients of x: b = 6a = 18

Comparing the constant terms: 10 = $9a + c$
i.e. c = 10 – 9a = 10 – 27 = -17.

So a = 3, b = 18 and c = -17.

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4 (i) $10^p = 0.1 \quad \Rightarrow \quad p = -1$

(Remember that a power of -1 means the reciprocal so $10^{-1} = 1/10$)

(ii) A power of ½ means the square root.

So $\left(25k^2\right)^{1/2} = \sqrt{25k^2} = 5k$

So, we need to solve 5k = 15 i.e. k = 3.

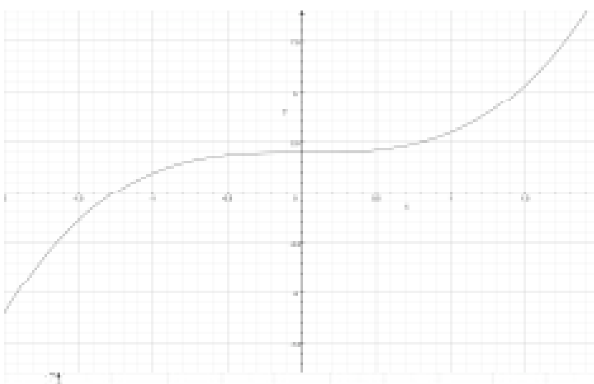
(iii) $t^{-1/3} = \frac{1}{t^{1/3}} = \frac{1}{\sqrt[3]{t}}$

So we need to solve

$$\frac{1}{\sqrt[3]{t}} = \frac{1}{2}$$

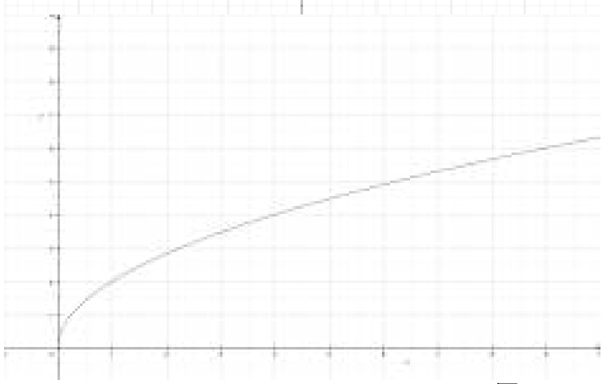
i.e. $\sqrt[3]{t} = 2 \quad \Rightarrow \quad t = 8$

5 (i)



The graph of $y = x^3 + 2$ is formed by translating the graph of $y = x^3$ by two units upwards.

(ii)



The graph of $y = 2\sqrt{x}$ is formed by stretching the graph of $y = \sqrt{x}$ in the direction of the y-axis by a scale factor of 2.

(iii) The transformation that transforms $y = 2\sqrt{x}$ onto the curve $y = 3\sqrt{x}$ is a stretch parallel to the y-axis scale factor 1.5.

6 (i) The solutions of the quadratic equation $ax^2 + bx + c = 0$ can be found using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

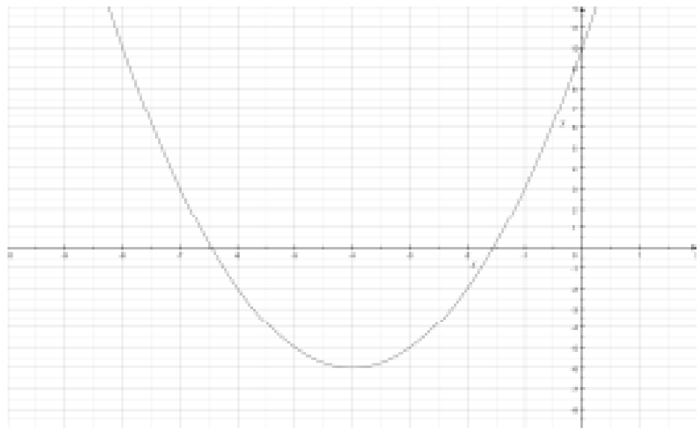
Here a = 1, b = 8 and c = 10.

So $x = \frac{-8 \pm \sqrt{64 - 4 \times 1 \times 10}}{2} = \frac{-8 \pm \sqrt{24}}{2}$

But $\sqrt{24} = \sqrt{4 \times 6} = 2\sqrt{6}$

Therefore the solutions are $x = \frac{-8 \pm 2\sqrt{6}}{2} = -4 \pm \sqrt{6}$

(ii) The curve cuts the y-axis at the point (0, 10).
The curve is a happy graph and cuts the x-axis at
$-4+\sqrt{6},\ -4-\sqrt{6}$ (both of which are negative values).



(iii) From the sketch, the solutions of the inequality are:

$x \geq -4+\sqrt{6}$ or $x \leq -4-\sqrt{6}$

7 (i) To find the equation of the line, rearrange to the form y = mx + c:

x+ 2y = 4 i.e 2y = 4 – x i.e. y = 2 – 0.5x.

So the gradient is -0.5.

(ii) A parallel line will have the same gradient, so the parallel line will have gradient -0.5.

The equation of a line with gradient m which passes through the point (a, b) is:

$y - y_1 = m(x - x_1)$

So

$y - 5 = -0.5(x - 6)$

Multiply by 2:

$2y - 10 = -x + 6$

Rearrange to get:

$x + 2y - 16 = 0$

(iii) Rearrange the equation x + 2y = 4 to make y the subject: y = 2 – 0.5x.

Substitute this into the equation of the curve:

$2 - 0.5x = x^2 + x + 1$

Multiply by 2 to remove the fraction:

$4 - x = 2x^2 + 2x + 2$

Rearrange to make one side equal to 0:

$2x^2 + 3x - 2 = 0.$

This equation can be solved by factorising:

$$2x^2 + 4x - 1x - 2 = 0$$
$$2x(x+2) - 1(x+2) = 0$$
$$(x+2)(2x-1) = 0$$

Therefore x = -2 or x = 0.5.

If x = -2, y = 2 – 0.5(-2) = 3
If x = 0.5, y = 2 – 0.5(0.5) = 1.75

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8 (i) To find the coordinates of the stationary points, the steps are:

1) Differentiate the equation of the curve to get $\frac{dy}{dx}$
2) Find the x-coordinates of the stationary points by solving $\frac{dy}{dx}$=0;
3) Find the y-coordinates of each point using the equation of the curve.

Here $y = x^3 + x^2 - x + 3$
Therefore:

$$\frac{dy}{dx} = 3x^2 + 2x - 1$$

To find the coordinates of the stationary points we solve $3x^2 + 2x - 1$=0.
This can be solved by factorising:

$$3x^2 + 2x - 1 = 3x^2 + 3x - 1x - 1$$
$$= 3x(x + 1) – 1(x + 1)$$
$$= (3x – 1)(x + 1)$$

So the solutions are x = 1/3 or x = -1.

When x = $\frac{1}{3}$, $y = \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \frac{1}{3} + 3 = \frac{1}{27} + \frac{1}{9} - \frac{1}{3} + 2 = 1\frac{22}{27}$
When x = -1, $y = (-1)^3 + (-1)^2 - (-1) + 3 = 4$

The coordinates of the stationary points are (-1, 4) and $\left(\frac{1}{3}, 1\frac{22}{27}\right)$.

(ii) To decide whether the stationary points are maximums or minimums the steps are:

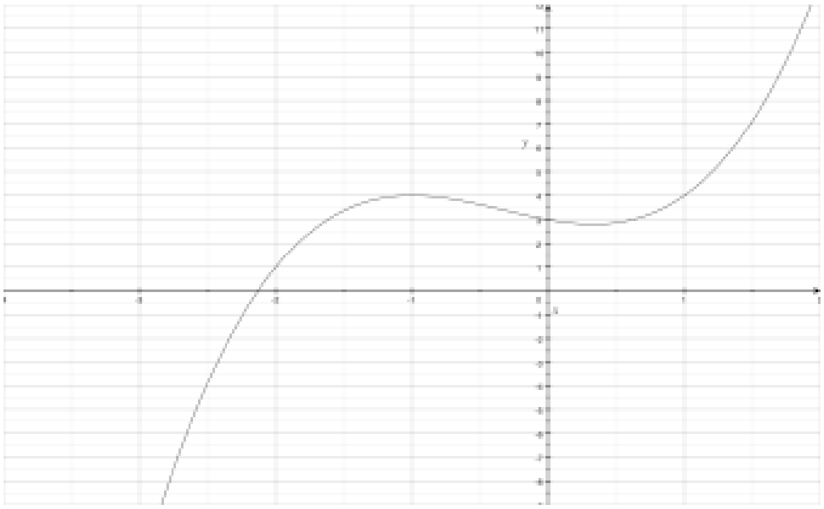
1) Find the second derivative $\frac{d^2y}{dx^2}$
2) Substitute each x value into the second derivative
3) If $\frac{d^2y}{dx^2} > 0$, then it is a minimum;
   If $\frac{d^2y}{dx^2} < 0$, then it is a maximum.

Here, $\frac{d^2y}{dx^2} = 6x + 2$

When x = -1, $\frac{d^2y}{dx^2} = 6(-1) + 2 = -4 < 0$, i.e. a maximum point

When x = 1/3, $\frac{d^2y}{dx^2} = 6(\frac{1}{3}) + 2 = 4 > 0$ i.e. a minimum point.

(iii) We can sketch the graph of $y = x^3 + x^2 - x + 3$:



This shows that the curve is decreasing if -1 < x < 1/3.

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9 (i) The gradient of the line AB is $m = \frac{y_2 - y_1}{x_2 - x_1}$ (LEARN THIS!)

So the gradient is $m = \frac{1-(-2)}{3-(-5)} = \frac{3}{8}$

Using the formula $y - y_1 = m(x - x_1)$ for the equation of a straight line, we get:

$$y - 1 = \tfrac{3}{8}(x - 3)$$

Multiply by 8 to remove the fraction:

$$8y - 8 = 3x - 9$$

Therefore: -3x + 8y + 1 = 0.

(ii) The coordinates of the midpoint of AB are: $\left(\frac{-5+3}{2}, \frac{-2+1}{2}\right) = (-1, -\tfrac{1}{2})$

(iii) We can calculate the distance between two points using Pythagoras's theorem (if we draw a sketch of the diagram) OR we can use the following formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

So: $d = \sqrt{(-3 - -5)^2 + (1 - -2)^2} = \sqrt{2^2 + 3^2} = \sqrt{13}$

(iv) Gradient of AC is $\frac{4-(-2)}{-3-(-5)} = \frac{6}{2} = 3$

Gradeint of BC is $\frac{4-(1)}{-3-3} = \frac{3}{-6} = -\frac{1}{2}$
AC is not perpendicular to BC as the product of their gradients is not -1.

(Note: Two lines are perpendicular if the product of their gradients is -1, i.e. if the gradient of one line is the negative reciprocal of the gradient of the other).

10(i) $$f(x) = 8x^3 + \frac{1}{x^3} = 8x^3 + x^{-3}$$ (Write as a negative power)

Therefore

$$f'(x) = 24x^2 + -3x^{-4} = 24x^2 - 3x^{-4}$$

(Remember the rule for differentiating: bring down the power and subtract one from the old power).

So

$$f''(x) = 48x + 12x^{-5} = 24x^2 - 3x^{-4}$$

(ii) We have to solve:

$$8x^3 + \frac{1}{x^3} = -9$$

Multiply by x^3:

$$8x^6 + 1 = -9x^3$$

So: $$8x^6 + 9x^3 + 1 = 0$$

This can be turned into a quadratic is we substitute $y = x^3$:

$$8y^2 + 9y + 1 = 0$$

This can be solved by factorising;

$$8y^2 + 8y + 1y + 1 = 8y(y+1) + 1(y+1) = (8y+1)(y+1)$$

So the solutions are y = -1 or $y = -\frac{1}{8}$
So:

$$x = \sqrt[3]{y} = \sqrt[3]{-1} = -1 \quad or \quad x = \sqrt[3]{-\tfrac{1}{8}} = -\tfrac{1}{2}$$