OCR Further Mathematics 1 (FP1)
Solutions: January 2007

1 (i) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} a & -1 \\ -3 & -2 \end{pmatrix}$

$2\mathbf{A} + \mathbf{B} = 2\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} a & -1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 4+a & 1 \\ 3 & 2 \end{pmatrix}.$

As the answer is $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$, we know that 4 + a = 1, i.e. that a = -3.

(ii) $\mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} a & -1 \\ -3 & -2 \end{pmatrix} = \begin{pmatrix} 2a-3 & -2-2 \\ 3a-6 & -3-4 \end{pmatrix} = \begin{pmatrix} 2a-3 & -4 \\ 3a-6 & -7 \end{pmatrix}$

As the answer is $\begin{pmatrix} 7 & -4 \\ 9 & -7 \end{pmatrix}$, we know that $2a - 3 = 7$, i.e. $a = 5$.

This value also makes the bottom left entry correct.

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2 Let $a + bi$ be a square root of 15 + 8i.

Then $(a+bi)^2 = 15 + 8i$

So, $a^2 + 2abi - b^2 = 15 + 8i$ (using $i^2 = -1$).

Comparing the imaginary parts, we have $2ab = 8$, i.e. $ab = 4$.

Comparing the real parts, we have $a^2 - b^2 = 15$.

As $b = 4/a$, we can write this as: $a^2 - \frac{16}{a^2} = 15$

Multiplying through by a^2 gives: $a^4 - 16 = 15a^2$ or $a^4 - 15a^2 - 16 = 0$.

This factorises: $(a^2 - 16)(a^2 + 1) = 0$

The solutions of this are $a^2 = 16$ i.e. $a = \pm 4$

or $a^2 = -1$ (this has no real solutions)

If $a = 4$, then $b = 4/4 = 1$

If $a = -4$, then $b = 4/-4 = -1$.

So the square roots are 4 + i and -4 – i.

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3 The standard results are: $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ (LEARN) and $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$ (IN FORMULA BOOK).

$$\sum_{r=1}^{n} r(r-1)(r+1) = \sum_{r=1}^{n} r(r^2-1) = \sum_{r=1}^{n}(r^3 - r)$$

$$= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r$$

$$= \tfrac{1}{4}n^2(n+1)^2 - \tfrac{1}{2}n(n+1)$$

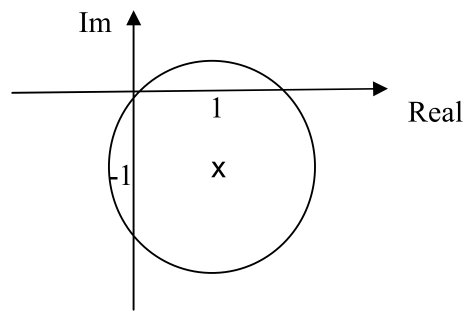
$$= \tfrac{1}{4}n^2(n+1)^2 - \tfrac{2}{4}n(n+1) \quad \text{(changing to a common denominator)}$$

Taking out a common factor,

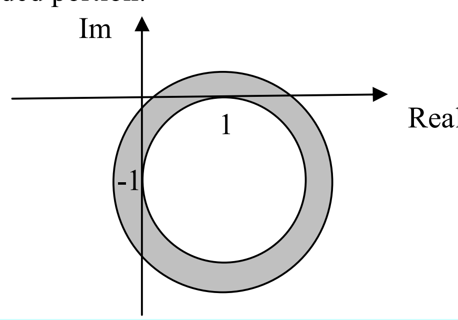
$$\sum_{r=1}^{n} r(r-1)(r+1) = \tfrac{1}{4}n(n+1)\big[n(n+1)-2\big]$$
$$= \tfrac{1}{4}n(n+1)\big(n^2+n-2\big)$$
$$= \tfrac{1}{4}n(n+1)(n+2)(n-1)$$

4 (i) $|z - 1 + i| = \sqrt{2}$ can be expressed as $|z - (1 - i)| = \sqrt{2}$

This represents a circle, centre $1 - i$, radius $\sqrt{2}$. (Note that this circle passes through the origin!)



(ii) The required region is shown below as the shaded portion:



5 (i) Expand out the RHS: $(z - 2)(z^2 + 2z + 4) = z^3 + 2z^2 + 4z - 2z^2 - 4z - 8$
$= z^3 - 8$ (as required)

(ii) We can use the quadratic formula to solve the equation $z^2 + 2z + 4 = 0$:

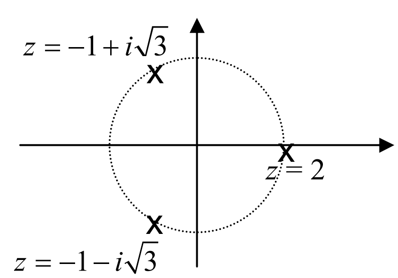
$$z = \frac{-2 \pm \sqrt{4 - 4\times 1\times 4}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$

These roots can be expressed as:

$$z = \frac{-2 \pm \sqrt{-1}\times\sqrt{12}}{2} = \frac{-2 \pm 2\sqrt{3}i}{2}$$

So the roots are $z = -1 + i\sqrt{3}$, $z = -1 - i\sqrt{3}$

(iii) The third root is $z = 2$.
The Argand diagram showing the three roots is:



6 (i) If $u_n = n^2 + 3n$, then $u_{n+1} = (n+1)^2 + 3(n+1) = (n^2 + 2n + 1) + (3n + 3) = n^2 + 5n + 4$.
Therefore,

$$u_{n+1} - u_n = n^2 + 5n + 4 - n^2 - 3n = 2n + 4$$

(ii) Step 1: We need to show that u_1 is divisible by 2. But $u_1 = 1^2 + 3(1) = 4$, which is a multiple of 2.

Step 2: We now assume that u_k is divisible by 2, i.e. that $u_k = 2F(k)$
We need to prove that u_{k+1} is also divisible by 2.
From part (i), we know that $u_{k+1} = 2k + 4 + u_k$.
This can be expressed as: $u_{k+1} = 2(k+2) + 2F(k) = 2[k + 2 + F(k)]$.
This proves that u_{k+1} is divisible by 2.

So by induction, each term in the sequence is divisible by 2.

7 (i) For any quadratic equation, $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

For the equation $x^2 + 5x + 10 = 0$, $\alpha + \beta = -\dfrac{5}{1} = -5$ and $\alpha\beta = \dfrac{10}{1} = 10$.

(ii) $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$.
Therefore, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-5)^2 - 2 \times 10 = 5$ (as required)

(iii) Sum of roots = $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{5}{10} = \dfrac{1}{2}$
Product of the roots = 1.

So the quadratic equation is:

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

This is

$$x^2 - \frac{1}{2}x + 1 = 0$$

or $2x^2 - x + 2 = 0$

8 (i) We can use the fact that $(r+2)! = (r+2)(r+1)!$
Therefore,

$$(r+2)! - (r+1)! = (r+2)(r+1)! - (r+1)!$$
$$= (r+1)!\left[r + 2 - 1\right]$$
$$= (r+1)! \times (r+1)$$

So, as $(r+1)! = (r+1)(r)!$ we get:

$$(r+2)! - (r+1)! = (r+1)^2 \times r!$$

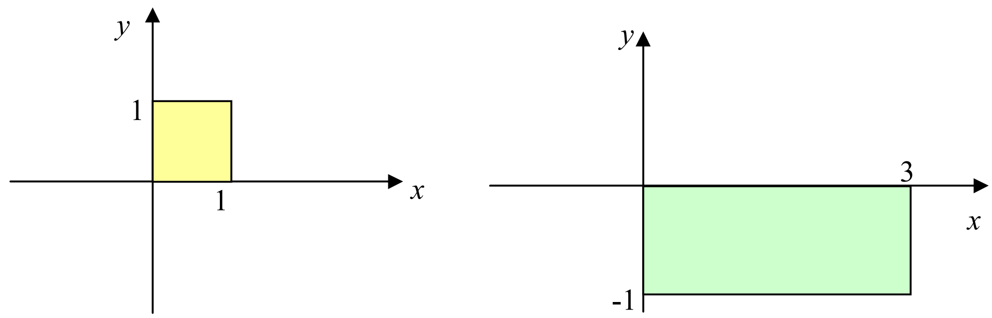
(ii) $2^2 \times 1! + 3^2 \times 2! + \ldots + (n+1)^2 \times n! = \sum_{r=1}^{n} (r+1)^2 \times r! = \sum_{r=1}^{n} (r+2)! - (r+1)!$

=(~~3!~~ - 2!) + (~~4!~~ – ~~3!~~) + (~~5!~~ – ~~4!~~) + …+ [(*n* + 2)! – ~~(*n* + 1)!~~]

= (*n* + 2)! – 2!

(iii) The series will not converge as the difference between the terms in the sequence gets larger and larger.

9 (i)



(ii) R is a rotation through 90 degrees in a clockwise direction, centre the origin. The matrix is:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(iii) S is a stretch in the *x* direction, scale factor 3. Its matrix is:

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

10 (i) To find the inverse of a 3 by 3 matrix, there are several steps:

Step 1: Find the determinant:

$$\text{Det}(\mathbf{D}) = a(1+2) - 2(3-0) + 0 = 3a - 6$$

Step 2: Find the matrix of cofactors:

$$\begin{pmatrix} 3 & 3 & -3 \\ 2 & a & -a \\ 4 & 2a & a-6 \end{pmatrix}$$

Step 3: Change the signs of every other element:

$$\begin{pmatrix} 3 & -3 & -3 \\ -2 & a & a \\ 4 & -2a & a-6 \end{pmatrix}$$

Step 4: Find the transpose:

$$\begin{pmatrix} 3 & -2 & 4 \\ -3 & a & -2a \\ -3 & a & a-6 \end{pmatrix}$$

Step 5: Divide by the determinant to get the inverse:

$$\mathbf{D}^{-1} = \frac{1}{3a-6}\begin{pmatrix} 3 & -2 & 4 \\ -3 & a & -2a \\ -3 & a & a-6 \end{pmatrix}$$

(ii) The equations can be expressed as:

$$\mathbf{D}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}.$$

Therefore, the solutions are:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{D}^{-1}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = \frac{1}{3a-6}\begin{pmatrix} 3 & -2 & 4 \\ -3 & a & -2a \\ -3 & a & a-6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

So,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3a-6}\begin{pmatrix} 9-8+4 \\ -9+4a-2a \\ -9+4a+a-6 \end{pmatrix} = \frac{1}{3a-6}\begin{pmatrix} 5 \\ 2a-9 \\ 5a-15 \end{pmatrix}$$