Further Pure 1
January 2006: Solutions

1 (i) $(1+8i)(2-i) = 2 - i + 16i - 8i^2 = 2 + 15i + 8 = 10 + 15i$

(ii) $\frac{1+8i}{2+i} = \frac{1+8i}{2+i} \times \frac{2-i}{2-i} = \frac{10+15i}{4-i^2}$ (using (i))

$$= \frac{10+15i}{5} = 2 + 3i$$

2

Case 1: $n = 1$ LHS $= \sum_{r=1}^{1} r^2 = 1^2 = 1$

RHS $= \frac{1}{6} \times 1 \times 2 \times 3 = 1$

Inductive step: Assume true when $n = k$, i.e. $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$

We need to prove statement true when $n = k + 1$, i.e. $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1)$

$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

To prove this:

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \quad \text{(from the assumption)}$$

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right]$$

$$= \frac{1}{6}(k+1)\left[2k^2 + k + 6k + 6\right] = \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) \quad \text{(as required)}$$

So the statement is true for $n = k + 1$.
Therefore by induction the statement is true for all positive integers n.

3 (i) $\det\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} = 2\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 2(6 - 1) - 1(3 - 1) + 3(1 - 2)$

$$= 10 - 2 - 3$$
$$= 5$$

(ii) M is non-singular since the determinant is non-zero.

4 $x = u + 2$ so $x^2 = u^2 + 4u + 4$

and $x^3 = (u^2 + 4u + 4)(u + 2) = u^3 + 6u^2 + 12u + 8$.

Substitute this into $x^3 - 6x^2 + 12x - 13 = 0$ gives

$$u^3 + 6u^2 + 12u + 8 - 6(u^2 + 4u + 4) + 12(u + 2) - 13 = 0$$

i.e. $u^3 - 5 = 0$

So $u = \sqrt[3]{5}$

Therefore $x = u + 2 = \sqrt[3]{5} + 2$

5

$$\sum_{r=1}^{n}(8r^3 - 6r^2 + 2r) = 8\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$= 8 \times \frac{1}{4}n^2(n+1)^2 - 6 \times \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1)$$

$$= 2n^2(n+1)^2 - n(n+1)(2n+1) + n(n+1)$$

$$= n(n+1)(2n(n+1) - (2n+1) + 1)$$

$$= n(n+1)(2n^2 + 2n - 2n - 1 + 1)$$

$$= n(n+1)(2n^2)$$

$$= 2n^3(n+1)$$

6 (i)

$$\mathbf{C}^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 8 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1.5 & 0.5 \end{pmatrix}$$

(ii) Method 1:

$$\mathbf{C} = \mathbf{AB} \quad \Rightarrow \quad \mathbf{CB}^{-1} = \mathbf{A}$$

$$\Rightarrow \quad \mathbf{B}^{-1} = \mathbf{C}^{-1}\mathbf{A}$$

So $\mathbf{B}^{-1} = \begin{pmatrix} 4 & -1 \\ -1.5 & 0.5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -2.5 & 0 \end{pmatrix}$

Method 2:

$$\mathbf{B} = \mathbf{A}^{-1}\mathbf{C} = \frac{1}{5}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 0 & -2 \\ 5 & 14 \end{pmatrix} = \begin{pmatrix} 0 & -0.4 \\ 1 & 2.8 \end{pmatrix}$$

Therefore

$$\mathbf{B}^{-1} = \frac{1}{-(-0.4)}\begin{pmatrix} 2.8 & 0.4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -2.5 & 0 \end{pmatrix}$$

7 (a) w = 3 + 2i

(i) $|w| = \sqrt{(3^2 + 2^2)} = \sqrt{13}$

(ii) Arg(z) = $\tan^{-1}(2/3) = 0.588$

(b) Let $u = a + bi$.

Then $(a + bi) + 2(a - bi) = 3 + 2i$

So $3a - bi = 3 + 2i$

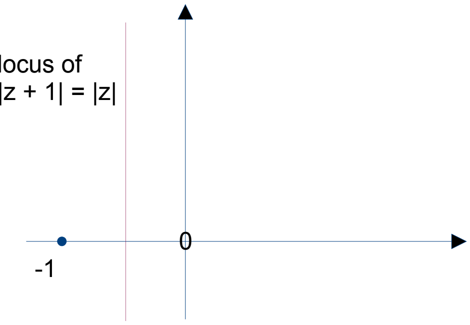
Therefore:

$3a = 3$ i.e. $a = 1$ (comparing real parts)

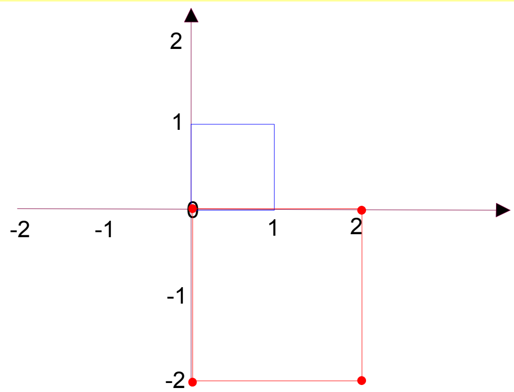
$-b = 2$ i.e. $b = -2$ (comparing imaginary parts)

So, $u = 1 - 2i$.

(c) The required locus is the perpendicular bisector of the line joining the complex numbers -1 and 0 on an Argand diagram:



8 (i) The image of (1, 0) is (2, 0)
The image of (0, 1) is (0, -2)
The image of (1, 1) is (2, -2)



(ii) To get from the unit square to its image under transformation T, two transformations are involved:

1) Enlargement scale factor 2, centre (0, 0).

The matrix for this is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

2) Reflection in the x-axis.

The matrix for this is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The transformations could be carried out in either order, so A could be either of the above so long as B is the other.

9 (i) $$\frac{1}{r} - \frac{1}{r+2} = \frac{r+2}{r(r+2)} - \frac{r}{r(r+2)} = \frac{r+2-r}{r(r+2)} = \frac{2}{r(r+2)}$$

(ii) Notice that $\frac{2}{1\times 3} + \frac{2}{2\times 4} + \ldots + \frac{2}{n(n+2)} = \sum_{r=1}^{n} \frac{2}{r(r+2)} = \sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+2}$.

Writing this in full, we get:

$$\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}$$

(iii) a) As n gets larger and larger, the value of $\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}$ gets closer and closer to $\frac{3}{2}$.

Therefore, $\sum_{r=1}^{\infty}\frac{2}{r(r+2)}=\frac{3}{2}$

b) $\sum_{r=n+1}^{\infty}\frac{2}{r(r+2)}=\sum_{r=1}^{\infty}\frac{2}{r(r+2)}-\sum_{r=1}^{n}\frac{2}{r(r+2)}=\frac{3}{2}-\left(\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$

$=\frac{1}{n+1}+\frac{1}{n+2}$

10 (i) $\alpha+\beta+\gamma=-\frac{b}{a}=-\frac{-9}{1}=9$

(ii) If $\beta=p+iq$, then $\gamma=p-iq$ (as the complex roots of an equation with real coefficients are complex conjugates of each other).
Since $\alpha+\beta+\gamma=9$, we have $\alpha+p+iq+p-iq=9$.
Therefore, $2p = 9 - \alpha$ or $p=\frac{9-\alpha}{2}$.

(iii) $\alpha\beta\gamma=-\frac{d}{a}=29$

(iv) Using the result from (iii), we have:

$$\alpha(p+iq)(p-iq)=29$$

Therefore,

$$\alpha(p^2+q^2)=29 \qquad \text{(remembering } i^2 = -1\text{)}$$

So $q^2=\frac{29}{\alpha}-p^2$

Using (ii), we get:

$$q^2=\frac{29}{\alpha}-\left(\frac{9-\alpha}{2}\right)^2$$

or $$q=\sqrt{\frac{29}{\alpha}-\left(\frac{9-\alpha}{2}\right)^2}$$