Further Pure 1
June 2005: Solutions

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$$\sum_1^n (6r^2+2r+1) = 6\sum_1^n r^2 + 2\sum_1^n r + \sum_1^n 1 \qquad \text{(split up sum)}$$

$$= 6\times\tfrac{1}{6}n(n+1)(2n+1) + 2\times\tfrac{1}{2}n(n+1) + n \quad \text{(using standard formulae)}$$

$$= n(n+1)(2n+1) + n(n+1) + n$$

$$= n\left[(n+1)(2n+1) + (n+1) + 1\right] \text{(factorising)}$$

$$= n\left[2n^2 + n + 2n + 1 + n + 1 + 1\right] \qquad \text{(expanding brackets)}$$

$$= n\left[2n^2 + 4n + 3\right]$$

2 (i)

$$\mathbf{A}^2 = \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}^2 = \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} = \begin{pmatrix}3 & 8\\ 4 & 11\end{pmatrix}$$

$$4\mathbf{A} - \mathbf{I} = 4\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}4 & 8\\ 4 & 12\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}3 & 8\\ 4 & 11\end{pmatrix}$$

Therefore $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$.

(ii) Method 1: Rearranging $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$, we get $4\mathbf{A} - \mathbf{A}^2 = \mathbf{I}$

Therefore $\mathbf{A}(4\mathbf{I} - \mathbf{A}) = \mathbf{I}$ (factorising)

So since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, we must have $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$.

Method 2: $\mathbf{A}^{-1} = \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix} = \begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}$

Also $4\mathbf{I} - \mathbf{A} = 4\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} = \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix} - \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} = \begin{pmatrix}3 & -2\\ -1 & 1\end{pmatrix}$

So $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$.

3 (i) $z + 5w = 2 + 3i + 5(4 - i) = 2 + 3i + 20 - 5i = 22 - 2i$

(ii) $z^* w = (2 - 3i)(4 - i) = 8 - 2i - 12i + 3i^2$

$= 8 - 14i - 3 \qquad (\text{since } i^2 = -1)$

$= 5 - 14i$

(iii) $\dfrac{1}{w} = \dfrac{1}{4-i} = \dfrac{1}{4-i}\times\dfrac{4+i}{4+i} = \dfrac{4+i}{16+4i-4i-i^2} =$

$= \dfrac{4+i}{17}$ or $\tfrac{4}{17} + \tfrac{1}{17}i$

4 Let the square root of $21 - 20i$ be $a + bi$.

Then, by definition of a square root, $(a+bi)^2 = 21-20i$.

So

$$\left(a+bi\right)\left(a+bi\right) = 21-20i$$

$$a^2 + 2abi - b^2 = 21-20i \qquad \text{(since } i^2 = -1\text{)}$$

Therefore

$$a^2 - b^2 = 21 \quad ① \qquad \text{(equating real parts)}$$

$$2ab = -20 \quad \Rightarrow \quad ab = -10 \quad ② \qquad \text{(coefficients of } i\text{)}$$

From equation ②, we know $b = -\frac{10}{a}$. Substituting this into ①, we get:

$$a^2 - \left(-\frac{10}{a}\right)^2 = 21$$

i.e. $a^2 - \left(\frac{100}{a^2}\right) = 21$

i.e. $a^4 - 100 = 21a^2$

So $a^4 - 21a^2 - 100 = 0$

$\Rightarrow$ $(a^2 - 25)(a^2 + 4) = 0$

Either $a^2 = 25$ or $a^2 = -4$.

But a is real, so we must have $a^2 = 25$, i.e. $a = 5$ or -5.

When $a = 5$, $b = -2$ and when $a = -5$, $b = 2$.

So the square roots are $5 - 2i$ and $-5 + 2i$.

5 (i) $$\frac{r+1}{r+2} - \frac{r}{r+1} = \frac{(r+1)(r+1)}{(r+2)(r+1)} - \frac{r(r+2)}{(r+2)(r+1)} \qquad \text{(writing with common denominator)}$$

$$= \frac{r^2 + 2r + 1 - (r^2 + 2r)}{(r+2)(r+1)}$$

$$= \frac{1}{(r+1)(r+2)}.$$

(ii) $$\tfrac{1}{6} + \tfrac{1}{12} + \tfrac{1}{20} + \dots + \tfrac{1}{(n+1)(n+2)} = \sum_{r=1}^{n} \tfrac{1}{(r+1)(r+2)} = \sum_{r=1}^{n} \tfrac{r+1}{r+2} - \tfrac{r}{r+1}$$

$$= \left(\cancel{\frac{2}{3}} - \frac{1}{2}\right) + \left(\cancel{\frac{3}{4}} - \cancel{\frac{2}{3}}\right) + \left(\cancel{\frac{4}{5}} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{5}{6}} - \cancel{\frac{4}{5}}\right) + \left(\cancel{\frac{5}{6}} - \dots + \left(\frac{n+1}{n+2} - \cancel{\frac{n}{n+1}}\right)\right.$$

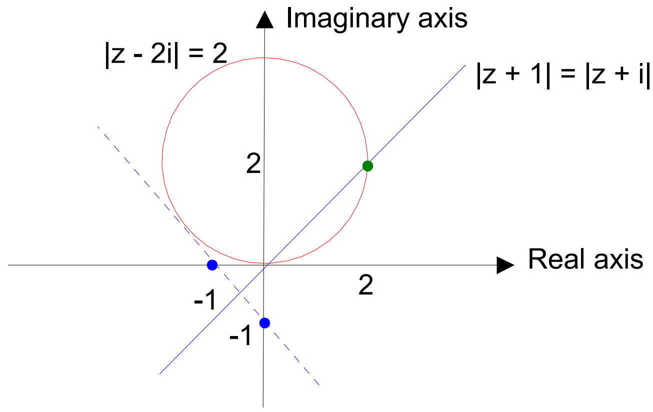
$$= \frac{n+1}{n+2} - \frac{1}{2}$$

(or $\frac{2n+1}{2(n+2)}$)

(iii) As n increases, $\frac{n+1}{n+2}$ gets closer and closer to 1.

So, $\sum_{r=1}^{\infty}\frac{1}{(r+1)(r+2)} = 1 - \frac{1}{2} = \frac{1}{2}$

6 (i)



(ii) From the diagram, the two loci intersect at 0 and $2 + 2i$.

7 (i)
$$\mathbf{B} = \begin{pmatrix} a & 1 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

If **B** is singular, then det(**B**) = 0.

$$\text{So, } \det(\mathbf{B}) = \det\begin{pmatrix} a & 1 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} = a\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= a(2+1) - 1(4-0) + 3(2-0)$$
$$= 3a - 4 + 6$$
$$= 3a + 2.$$

So if det(**B**) = 0, then $a = -\frac{2}{3}$.

(ii) From above, det(**B**) = $3a + 2$.

The matrix of minor determinants is $\begin{pmatrix} 3 & 4 & 2 \\ -1 & 2a & a \\ -4 & -a-6 & a-2 \end{pmatrix}$.

Adjust signs of every other element: $\begin{pmatrix} 3 & -4 & 2 \\ 1 & 2a & -a \\ -4 & a+6 & a-2 \end{pmatrix}$.

Take the transpose and divide by the determinant: $\frac{1}{3a+2}\begin{pmatrix} 3 & 1 & -4 \\ -4 & 2a & a+6 \\ 2 & -a & a-2 \end{pmatrix}$.

(iii) The equations can be written in matrix form as: $\begin{pmatrix} -1 & 1 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$.

This corresponds to the matrix in (i) with a = -1.
The solutions can be obtained using the inverse matrix:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-3+2}\begin{pmatrix} 3 & 1 & -4 \\ -4 & -2 & 5 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 11 \\ -17 \\ 9 \end{pmatrix} = \begin{pmatrix} -11 \\ 17 \\ -9 \end{pmatrix}$$

Therefore,

$x = -11,\ y = 17,\ z = -9.$

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8 (a) (i) $\alpha + \beta = \frac{-(-2)}{1} = 2$ $\qquad \alpha\beta = \frac{4}{1} = 4$

(ii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = (2)^2 - 2(4) = -4$.

(iii) $\alpha^2\beta^2 = (\alpha\beta)^2 = 4^2 = 16$

A quadratic equation can be expressed as $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

So, $x^2 - (-4)x + (16) = 0$

i.e. $x^2 + 4x + 16 = 0$

(b) (i) Sum of roots = $p + 2p + 3p = 6p$

But sum of roots = $\frac{-b}{a} = \frac{12}{1} = 12$.

Therefore p = 2.

(ii) The 3 roots are 2, 4 and 6.

Using the result $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, we see that $2\times4 + 2\times6 + 4\times6 = \frac{a}{1}$.

Therefore 44 = a.

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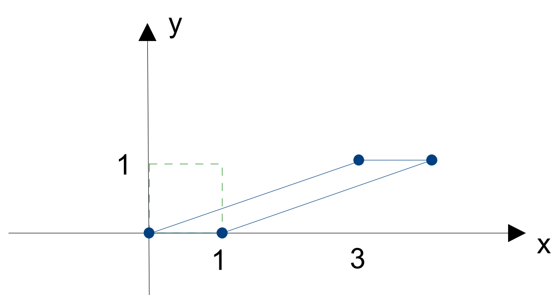
9 (i) The required matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii) $\mathbf{D} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(1, 0) maps to (1, 0)

(0, 1) maps to (3, 1).



So we see that the matrix represents a **shear** with the x-axis as the invariant line and such that (0, 1) maps to (3, 1).

(iii) $\mathbf{M} = \mathbf{DC} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ (as required)

(iv) $\mathbf{M}^1 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^1 & 3\left(2^1 - 1\right) \\ 0 & 1 \end{pmatrix}$

So the result is true when n = 1.

Inductive step: Assume the result is true when $n = k$, i.e. $\mathbf{M}^k = \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$.

We need to prove the result is true when $n = k + 1$, i.e. that $\mathbf{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$

But $\mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M} = \begin{pmatrix} 2^k & 3(2^k - 1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 3 \times 2^k + 3(2^k - 1) \\ 0 & 1 \end{pmatrix}$

Expanding out the entry in the top right hand corner:

$$\begin{aligned} 3 \times 2^k + 3(2^k - 1) &= 3 \times 2^k + 3 \times 2^k - 3 = 6 \times 2^k - 3 \\ &= 3 \times 2 \times 2^k - 3 \\ &= 3(2^{k+1} - 1) \end{aligned}$$

So, $\mathbf{M}^{k+1} = \begin{pmatrix} 2^{k+1} & 3(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$.

Therefore the result is true for $n = k + 1$.
Therefore the result is true for all positive integers n.