OCR Further Pure Mathematics 1 (FP1)
Solutions – June 2006

1 (i)
$$\mathbf{A} + 3\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} + 3\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 3 \\ 0 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & 4 \\ 0 & -1 \end{pmatrix}$$

(ii)
$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$
$$= 3\mathbf{I}.$$

So $k = 3$.

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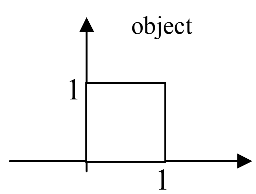
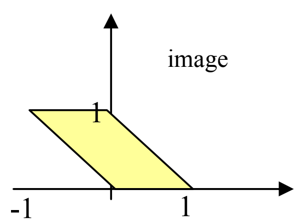
2 (i) The shear is parallel to the x-axis so points on this axis do not move. Therefore the image of (1, 0) is (1, 0).





(ii) Recall that the first column of a transformation matrix is the image of the point (1, 0) whilst the second column is the image of the point (0, 1).

The matrix representing transformation S therefore is $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

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3 (i) The roots of a quadratic equation with real coefficients are complex conjugates of each other. Therefore the second root is 2 + 3i.

(ii) Method 1: Substitute one of the roots into the quadratic equation.
For example, if we put $x = 2 - 3i$, we get:

$$(2-3i)^2 + p(2-3i) + q = 0$$
$$(2-3i)(2-3i) + p(2-3i) + q = 0$$
$$(4 - 12i - 9) + p(2-3i) + q = 0 \qquad (\text{as } i^2 = -1)$$
$$(-5 - 12i) + p(2-3i) + q = 0$$

Take the real parts: $-5 + 2p + q = 0$ i.e. $2p + q = 5$

Take the imaginary parts: $-12 - 3p = 0$

i.e. $3p = -12$

i.e. $p = -4$

Using the first equation, we therefore get $q = 5 - 2p = 13$

Method 2: We know that the sum of the roots is given by the formula

$$\alpha + \beta = -\frac{b}{a}$$

As the roots are 2 – 3i and 2 + 3i we therefore have:

$$(2 - 3i) + (2 + 3i) = -p.$$

This leads to $p = -4$.

The product of the roots is given by the result: $\alpha\beta = \frac{c}{a}$.

Therefore, $(2 - 3i)(2 + 3i) = q$

This leads to $q = 13$.

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4 From the formula book, the relevant formulae are:

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{6}n(n+1)(2n+1) \quad \text{and} \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{4}n^2(n+1)^2$$

So:

$$\sum_{r=1}^{n}(r^3 + r^2) = \sum_{1}^{n} r^3 + \sum_{1}^{n} r^2$$

$$= \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{4}n^2(n+1)^2$$

$$= \tfrac{2}{12}n(n+1)(2n+1) + \tfrac{3}{12}n^2(n+1)^2 \quad \text{(changing to common denom.)}$$

$$= \tfrac{1}{12}n(n+1)[2(2n+1) + 3n(n+1)]$$

$$= \tfrac{1}{12}n(n+1)\left[4n + 2 + 3n^2 + 3n\right]$$

$$= \tfrac{1}{12}n(n+1)\left[3n^2 + 7n + 2\right]$$

$$= \tfrac{1}{12}n(n+1)(n+2)(3n+1)$$

as required.

5(i) $2z - 3w = 2(3-2i) - 3(2+i) = 6 - 4i - 6 - 3i = -7i$

(ii) $iz = i(3-2i) = 3i + 2 \quad (\text{as} \quad i^2 = -1)$

Therefore

$$(iz)^2 = (2+3i)^2 = (2+3i)(2+3i)$$

$$= 4 + 12i - 9$$

$$= -5 + 12i$$

(iii) $\frac{z}{w} = \frac{3-2i}{2+i} = \frac{3-2i}{2+i} \times \frac{2-i}{2-i}$ (multiply top and bottom by complex conjugate of w)

$$= \frac{6 - 3i - 4i - 2}{4 - 2i + 2i + 1}$$

$$= \frac{4 - 7i}{5}$$

$$= \frac{4}{5} - \frac{7}{5}i$$

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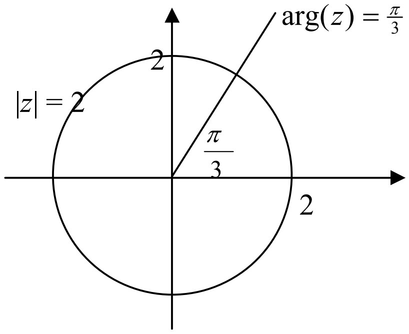
6 (i)



(ii) We can use SOHCAHTOA to find the coordinates of intersection:

$$x = 2\cos\tfrac{\pi}{3} = 1$$

$$y = 2\sin\tfrac{\pi}{3} = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the intersection is $1+\sqrt{3}i$

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7 (i) $\mathbf{A}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{A}^3 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) These results suggest that $\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 0 & 1 \end{pmatrix}$

(iii) The result is clearly true when $n = 1$ as $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

Assume the result is true when $n = k$, i.e. $\mathbf{A}^k = \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix}$

We need to prove the result is true when $n = k + 1$, i.e that $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2^{k+1} & 0 \\ 0 & 1 \end{pmatrix}$.

Starting from the left hand side:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{k}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{as we are assuming result true when } n = k$$

$$= \begin{pmatrix} 2^k\times 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{as required.}$$

Therefore the result is true for $n = k + 1$.
Therefore by induction the result is true for all positive integers n.

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8 (i)

The determinant of a general 3 by 3 matrix $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ is:

$$a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

So here

$$\det\begin{pmatrix} a & 4 & 2 \\ 1 & a & 0 \\ 1 & 2 & 1 \end{pmatrix} = a\begin{vmatrix} a & 0 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & a \\ 1 & 2 \end{vmatrix}$$

$$= a(a-0) - 4(1-0) + 2(2-a)$$
$$= a^2 - 4 + 4 - 2a$$
$$= a^2 - 2a$$

(ii) As matrix is singular if it has zero determinant.
So here **M** is singular if $a^2 - 2a = 0$
i.e. if $a(a-2) = 0$
i.e. if $a = 0$ or $a = 2$.

(iii)

The equations can be written as $\mathbf{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3a \\ 1 \\ 3 \end{pmatrix}$.

a) If $a = 3$, the matrix **M** can be inverted. So the equations will have a unique solution.

b) If $a = 2$, the matrix **M** is singular. So either there are no solutions or there are an infinite number of solutions.
To decide which situation we are in, we need to try to solve the equations:

$$\begin{aligned} 2x + 4y + 2z &= 6 \quad (1) \\ x + 2y \quad\quad\;\; &= 1 \quad (2) \\ x + 2y + z &= 3 \quad (3) \end{aligned}$$

Notice that equation 1 is double equation 3.
So we really only have 2 independent equations.
There is therefore an infinite number of equations.

9 (i)

$$\sum_{r=1}^{n}\left\{(r+1)^3 - r^3\right\} = \underbrace{(2^3 - 1^3)}_{r=1} + \underbrace{(3^3 - 2^3)}_{r=2} + \underbrace{(4^3 - 3^3)}_{r=3} + \ldots + \underbrace{(n^3 - (n-1)^3)}_{r=n-1} + \underbrace{((n+1)^3 - n^3)}_{r=n}$$

$$= (n+1)^3 - 1$$

(ii) $(r+1)^3 = (r^2 + 2r + 1)(r + 1) = r^3 + 3r^2 + 3r + 1$
Therefore $(r + 1)^3 - r^3 = 3r^2 + 3r + 1$.

(iii)

$$\sum_{r=1}^{n}\left\{(r+1)^3 - r^3\right\} = \sum_{r=1}^{n}(3r^2 + 3r + 1) = 3\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 3\sum_{r=1}^{n} r^2 + 3 \times \tfrac{1}{2}n(n+1) + n$$

Using part (i), we therefore have:

$$3\sum_{r=1}^{n} r^2 + \tfrac{3}{2}n(n+1) + n = (n+1)^3 - 1$$

Therefore:

$$3\sum_{r=1}^{n} r^2 = (n+1)^3 - 1 - \tfrac{3}{2}n(n+1) - n$$

$$= (n^3 + 3n^2 + 3n + 1) - 1 - \tfrac{3}{2}n^2 - \tfrac{3}{2}n - n$$

$$= n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

$$= \tfrac{1}{2}n\left(2n^2 + 3n + 1\right)$$

$$= \tfrac{1}{2}n(n+1)(2n+1)$$

So

$$3\sum_{r=1}^{n} r^2 = \tfrac{1}{2}n(n+1)(2n+1) \qquad \text{as required}$$

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10 (i) $x^3 - 2x^2 + 3x + 4 = 0$ has roots $\alpha,\ \beta,\ \gamma$

$$\alpha + \beta + \gamma = -\frac{b}{a} = 2$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 3$$

$$\alpha\beta\gamma = -\frac{d}{a} = -4$$

(ii) Let $u = \alpha + 1$

(iii) As α is a root, $(\alpha)^3 - 2\alpha^2 + 3\alpha + 4 = 0$

Replace α by $u - 1$:

$$(u-1)^3 - 2(u-1)^2 + 3(u-1) + 4 = 0$$

$$(u^3 - 3u^2 + 3u - 1) - 2(u^2 - 2u + 1) + 3u - 3 + 4 = 0$$

i.e. $u^3 - 5u^2 + 10u - 2 = 0$

Therefore, the equation is:

$$x^3 - 5x^2 + 10x - 2 = 0$$

So p = -5 and q = -2.