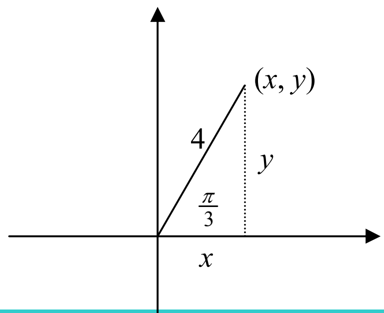
OCR Further Pure Mathematics 1
Solutions – June 2007

1 From the diagram:

$$x = 4\cos\tfrac{\pi}{3} = 2$$

$$y = 4\sin\tfrac{\pi}{3} = 2\sqrt{3}$$



2 $$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$$

Step 1: Prove true when $n = 1$:

LHS = $1^3 = 1$

RHS = $\tfrac{1}{4} \times 1^2 \times 2^2 = 1$

Therefore the result is true for $n = 1$.

Step 2: Assume the result is true for $n = k$, i.e. that $\sum_{r=1}^{k} r^3 = \tfrac{1}{4}k^2(k+1)^2$.

We need to prove the result is true for $n = k + 1$, i.e. that $\sum_{r=1}^{k+1} r^3 = \tfrac{1}{4}(k+1)^2(k+2)^2$.

$$LHS = \sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k+1)^3$$

$$= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 \qquad \text{(by inductive assumption)}$$

$$= \tfrac{1}{4}k^2(k+1)^2 + \tfrac{4}{4}(k+1)^3 \qquad \text{(change to common denominator)}$$

$$= \tfrac{1}{4}(k+1)^2[k^2 + 4(k+1)]$$

$$= \tfrac{1}{4}(k+1)^2(k+2)^2$$

$$= RHS$$

So the result is true when $n = k + 1$.

Therefore, by induction, the result is true for all positive integers n.

3 The standard results are:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1) \qquad \text{(Learn this - not in formula book)}$$

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1) \qquad \text{(In formula book)}$$

So, using these, we get:

$$\sum_{r=1}^{n}(3r^2-3r+1) = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= 3\times\tfrac{1}{6}n(n+1)(2n+1) - 3\times\tfrac{1}{2}n(n+1) + n$$

$$= \tfrac{1}{2}n(n+1)(2n+1) - \tfrac{3}{2}n(n+1) + \tfrac{2}{2}n$$

Taking out the common factor we get:

$$\sum_{r=1}^{n}(3r^2-3r+1) = \tfrac{1}{2}n\left[(n+1)(2n+1) - 3(n+1) + 2\right]$$

$$= \tfrac{1}{2}n\left[2n^2+3n+1-3n-3+2\right]$$

$$= \tfrac{1}{2}n\left[2n^2\right]$$

$$= n^3$$

4 (i) $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix} \quad \Rightarrow \quad \det(\mathbf{A}) = 1\times 5 - 1\times 3 = 2$

So, $\mathbf{A}^{-1} = \tfrac{1}{2}\begin{pmatrix} 5 & -1 \\ -3 & 1 \end{pmatrix}$

(ii) The simplest way to work out $(\mathbf{AB})^{-1}$ is to use this useful result:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Applying this here, we get:

$$(\mathbf{AB})^{-1} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\tfrac{1}{2}\begin{pmatrix} 5 & -1 \\ -3 & 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 5 & -1 \\ -3 & 1 \end{pmatrix}$$

$$= \tfrac{1}{2}\begin{pmatrix} 2 & 0 \\ 23 & -5 \end{pmatrix}$$

5 (i) $\frac{1}{r} - \frac{1}{r+1} = \frac{r+1}{r(r+1)} - \frac{r}{r(r+1)} = \frac{1}{r(r+1)}$ (as required)

(ii) $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)} = \sum_{r=1}^{n}\frac{1}{r(r+1)}$

$$= \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) \qquad \text{(from (i))}$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

(iii) $\sum_{r=1}^{\infty} \frac{1}{r(r+1)} = 1$ (from part (ii))

So,

$$\sum_{r=n+1}^{\infty} \frac{1}{r(r+1)} = \sum_{r=1}^{\infty} \frac{1}{r(r+1)} - \sum_{r=1}^{n} \frac{1}{r(r+1)}$$
$$= 1 - \left(1 - \frac{1}{n+1}\right)$$
$$= \frac{1}{n+1}$$

---

6 (i) $3x^3 - 9x^2 + 6x + 2 = 0$

a) Sum of roots: $\alpha + \beta + \gamma = -\frac{b}{a} = -\frac{-9}{3} = 3$

Sum of pairs of products: $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = \frac{6}{3} = 2$

b) $(\alpha + \beta + \gamma)^2 = (\alpha + \beta + \gamma)(\alpha + \beta + \gamma) = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma$

Therefore,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma$$
$$= 3^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$
$$= 3^2 - 2(2)$$
$$= 5$$

(ii) a) Replacing x by $\frac{1}{u}$ in the original cubic equation results in

$$3\left(\frac{1}{u}\right)^3 - 9\left(\frac{1}{u}\right)^2 + 6\left(\frac{1}{u}\right) + 2 = 0$$

Multiplying by u^3, gives

$$3 - 9u + 6u^2 + 2u^3 = 0$$

or $$2u^3 + 6u^2 - 9u + 3 = 0$$

b) The roots of this new equation will be $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

Therefore $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ represents the sum of the roots, i.e. $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{-6}{2} = -3$

---

7 (i)

The determinant of the 3 by 3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$.

So the determinant of $\mathbf{M} = \begin{pmatrix} a & 4 & 0 \\ 0 & a & 4 \\ 2 & 3 & 1 \end{pmatrix}$ is

$$\det(\mathbf{M}) = a\begin{vmatrix} a & 4 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 0 & 4 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & a \\ 2 & 3 \end{vmatrix}$$
$$= a(a-12) - 4(0-8) + 0$$
$$= a^2 - 12a + 32$$

(ii) If $a = 2$, $\det(\mathbf{M}) = 2^2 - 12(2) + 32 \neq 0$, so **M** is not singular.

(iii) If $a = 4$, $\det(\mathbf{M}) = 4^2 - 12(4) + 32 = 0$, so **M** is singular.
The equations are:

$$4x + 4y = 6$$
$$4y + 4z = 8$$
$$2x + 3y + z = 1$$

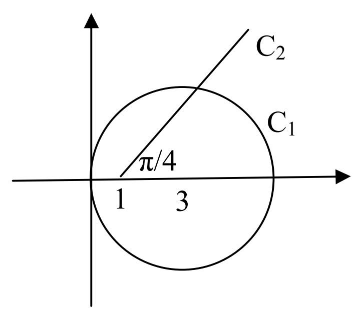
These equations simplify to

$$2x + 2y = 3$$
$$y + z = 2$$
$$2x + 3y + z = 1$$

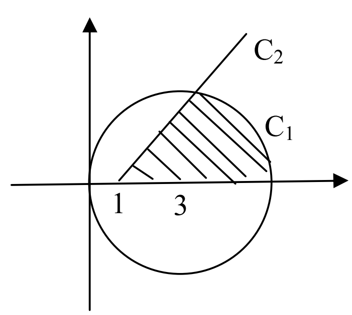
Eliminate z from the second and third equations by subtracting:
$2x + 2y = -1$
The first equation was $2x + 2y = 3$.
The equations are therefore inconsistent and so have no solutions.

8 (i) $|z - 3| = 3$ represents a circle centre (3, 0) radius 3.
$\arg(z - 1) = \pi/4$ is a half-line starting at the point (1, 0) making an angle of $\pi/4$ with the horizontal.

The Argand diagram therefore is:



(ii) The required region is shaded on this diagram:



9 (i) The matrix for an enlargement centre (0, 0) scale factor √2 is

$$\mathbf{A} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

(ii) The matrix that represents a rotation through an angle of θ degrees anticlockwise is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

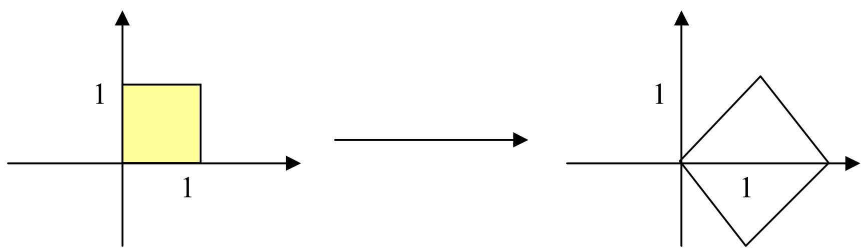
Here the matrix is $\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$.

Comparing the bottom left hand corner: $\sin\theta = -\frac{\sqrt{2}}{2}$

Using a calculator, the solution to this is θ = -45°.
So the matrix represents a rotation 45 degrees anticlockwise centre (0, 0).

(iii) $\mathbf{C} = \mathbf{AB} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

(iv)



(1, 0) is mapped to (1, -1)
(0, 1) is mapped to (1, 1)

(v) Det(**C**) = 1 – (-1) = 2
This shows that when a shape is enlarged by a transformation represented by the matrix **C**, the area of the image is twice the area of the object.

10 (i) Let $a + bi$ be a square root of 16 + 30i.
Then $(a+bi)^2 = 16 + 30i$
So, $a^2 + 2abi - b^2 = 16 + 30i$ (using $i^2 = -1$).

Comparing the imaginary parts, we have $2ab = 30$, i.e. $ab = 15$.

Comparing the real parts, we have $a^2 - b^2 = 16$.

As $b = 15/a$, we can write this as: $a^2 - \frac{225}{a^2} = 16$

Multiplying through by a^2 gives: $a^4 - 225 = 16a^2$ or $a^4 - 16a^2 - 225 = 0$.

This factorises: $(a^2 - 25)(a^2 + 9) = 0$

The solutions of this are $a^2 = 25$ i.e. $a = \pm 5$
or $a^2 = -9$ (this has no real solutions)

If $a = 5$, then $b = 15/5 = 3$
If $a = -5$, then $b = 15/-5 = -3$.

So the square roots are 5 + 3i and -5 – 3i.

(ii) Using the quadratic formula, the solutions to the equation $z^2 - 2z - (15 + 30i) = 0$ are

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)(15 + 30i)}}{2}$$
$$z = \frac{2 \pm \sqrt{4 + 60 + 120i}}{2}$$
$$z = \frac{2 \pm \sqrt{64 + 120i}}{2}$$
$$z = \frac{2 \pm \sqrt{4}\sqrt{16 + 30i}}{2}$$
$$z = \frac{2 \pm 2\sqrt{16 + 30i}}{2}$$
$$z = 1 \pm \sqrt{16 + 30i}$$

Using the answer from (i) for the square root of 16 + 30i, we see that the solutions to the quadratic equation are;

$z = 1 + 5 + 3i = 6 + 3i$ or $z = 1 - (5 + 3i) = -4 - 3i$