Schoolworkout Maths

S1 Revision Notes: Arrangements

Section 1: Permutations

Recap:
- Permutations are ordered arrangements.
- The number of permutations of n unlike objects is $n! = n \times (n - 1) \times \ldots \times 2 \times 1$.
- The number of permutations of n objects, where a are of one kind, b are of a second kind, c are of a third kind, … is
$$\frac{n!}{a!b!c!\ldots}.$$
- The number of permutations of r objects from n unlike objects is ${}^nP_r = \frac{n!}{(n-r)!}$. There is a ${}^nP_r$ key on your calculator.

- **Example 1:**

A set of 8 square tiles, identical in every way except colour, are to be arranged in a straight line. Given that 3 are red, 3 are black and 2 are white, calculate
(i) the total number of different arrangements;
(ii) the number of arrangements in which the first and last tiles are white;
(iii) the number of arrangements in which the 2 white tiles have exactly 2 tiles, one red and the other black, between them.

- **Solution:**

(i) Number of arrangements = $\frac{8!}{3!3!2!} = 560$

(ii) The tiles must be placed like: W _ _ _ _ _ _ W.
There are 6 tiles that can be placed in the middle, 3 of which are red and 3 are black.
So number of arrangements = $\frac{6!}{3!3!} = 20$.

(iii) Consider first the case where the 2 white tiles have a red and a black (in that order between them). Stick these four tiles together to get $\boxed{\text{W R B W}}$.
There are now 5 tiles to be arranged (with 2 red and 2 black).
So number of arrangements is $\frac{5!}{2!2!} = 30$.

The white tiles could also be separated by the black and a red (in that order). This results in another 30 arrangements.

So total number of arrangements is 60

- **Example 2:**

a) How many three digit numbers can be formed using the digits 1, 2, 3, 4, 5, 6, 8 where each digit can be used only once?
b) One of the arrangements from (a) is picked at random. Find the probability it is odd?
c) How many of the arrangements from (a) are odd and over 500?

Solution:

(a) We are picking 3 digits from 7 different digits. This can be done is ${}^7P_3 = 210$ ways.

(We could also say… there are 7 choices for 1st digit
there are then 6 choices for 2nd digit
there are then 5 choices for 3rd digit
So $7 \times 6 \times 5 = 210$)

(b) There is now a restriction that the number must be odd, i.e. final digit must be 1, 3 or 5

So there are 3 choices for final digit
there are then 6 choices remaining for 1st digit
there are then 5 choices for 2nd digit.
So $3 \times 6 \times 5 = 90$.

Therefore probability is $\dfrac{90}{210} = \dfrac{3}{7}$.

(c) There are now restrictions on the first and third digits.
The first digit must be 5, 6, 8
The final digit must be 1, 3, 5

Note that the number 5 appears in both lists.

Case 1: Suppose 1st digit is 5.
There are then 2 possible values for 3rd digit
There are then 5 choices for 2nd digit.
So number of arrangements is $1 \times 2 \times 5 = 10$

Case 2: Suppose 1st digit is either 6 or 8.
There are then 2 choices for 1st digit
There are 3 choices for 3rd digit
There are 5 choices for 2nd digit.
So number of arrangements is $2 \times 3 \times 5 = 30$.

So altogether there are $10 + 30 = 40$ arrangements.

Section 2: Combinations

Recap:

- Combinations are arrangements where order is not important.
- The number of ways of selecting r objects from n unlike objects is ${}^nC_r = \frac{n!}{r!(n-r)!}$. There is a ${}^nC_r$ key on your calculator.

- **Example:**

A school has 4 vacancies for parent governors. 5 women and 3 men have applied to be a parent governor.
a) How many ways are there of selecting 4 people to fill the vacant posts?
b) If the 4 vacancies are filled at random, find the probability that there will be at least 2 female parent governors?
c) Two of the people applying to be governors are Mr and Mrs Smith. How many ways are there of filling the 4 vacancies if these two must both be chosen?

- **Solution:**

a) Number of possible selections of 4 people from 8 is ${}^8C_4$=70.

b) Number of ways with exactly 2 women (and therefore 2 men) = ${}^5C_2 \times {}^3C_2 = 10 \times 3 = 30$

Number of ways with 3 women (and 1 man) = ${}^5C_3 \times {}^3C_1 = 10 \times 3 = 30$

Number of ways with 4 women (and 0 men) = ${}^5C_4 = 5$
So number of ways with at least 2 women is 65.

So probability of at least 2 women = $\frac{65}{70} = \frac{13}{14}$.

c) If Mr and Mrs Smith must be both chosen then there are still 2 places to fill from 6 people. This can be done in ${}^6C_2 = 15$ ways.